# Always Check the Units Before Submitting Your Final Answer

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It had been quite a while since Anna had studied mathematics. She had enjoyed her math classes, and had been good at them, but she loved learning about lots of things, so since her last math class she had devoted her time to studying a wide range of other subjects.

Right now it looked like that decision had paid off, because she had been chosen to appear on a TV quiz show where you could win an absurd amount of money by answering questions about a vast assortment of different topics. Here she was, sitting in a brightly lit studio in front of the cameras, Nigel the host, and an excited studio audience. So far she had correctly answered questions about art, literature, history, psychology, cooking, medicine, sports, and music. She needed to answer just one more question to win the Grand Prize: an astounding one hundred thousand dollars.

The final question appeared on the screen:

If $ax^2 + bx + c = 0$, then the value of $x$ is given by...

(A) $x = \dfrac{-b \pm \sqrt{b - 4ac}}{2a}$

(B) $x = \dfrac{-b \pm \sqrt{c - 4b^2}}{2a}$

(C) $x = \dfrac{-c \pm \sqrt{c^2 - 4ac}}{2a}$

(D) $x = \dfrac{-c \pm \sqrt{b^2 - 4ac}}{2a}$

(E) $x = \dfrac{-b \pm \sqrt{b^2 - 4bc}}{2a}$

(F) $x = \dfrac{-b \pm \sqrt{4ac - ab^2}}{2a}$

(G) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(H) $x = \dfrac{-a \pm \sqrt{b^2 - 4ac}}{2a}$

It was the quadratic formula, of course. There was a time when she had known it by heart, but now all she could remember was the rhythm: "negative something plus or minus the square root of something something divided by something". Unfortunately, *all* the possible answers on the list matched that pattern. She knew there was a way to derive the formula by algebra—the phrase "completing the square" flitted through her mind—but that would require paper and a pencil or pen, none of which she had. Ditto for putting each of the possible answers into the equation to see which one worked. And the clock was ticking.

So, she thought, let's at least try a process of elimination: could any of the possible answers be ruled out?

It seemed obvious that all three of $a$, $b$, and $c$ would figure in the formula, so answer (C) was out. That one was a giveaway, for sure, but now what? Then she remembered a piece of advice from another class she had enjoyed: physics. *Always make sure that the units make sense before you submit your final answer.*

True, there were no units given in this problem, but maybe she could think of $x$ as a measure of something simple and see what happened. So, she thought, let's imagine that $x$ is a length. Then $x^2$ is the area of a square with side length $x$, and $ax^2$ is the total area of $a$ squares all with side length $x$. This interpretation makes the quadratic equation into an equation about areas. That means that $b$ must be a measure of length, so that $bx$ will be the area of a rectangle, and $c$ by itself must be an area of... well, something.

Now Anna needed to check each possible answer to see if it gave a value for $x$ that would be a length. Since $a$ was a number that told you how many squares or parts of squares you had—did they call that "dimensionless"?—the denominator $2a$ was also just a number, and so the numerator had to be a length. That meant that both of the terms in the numerator had to be lengths, too, because you can only add quantities with the same units. This ruled out (D), since $-c$ on top was an area, without even worrying about what was added to it. The same reasoning also eliminated (H): she had decided that the $a$ in the denominator was dimensionless, so it could not somehow become a length in the numerator.

What about the stuff under the radical? Well, since the square root itself had to give a length, the expression under the radical had to be the area of a square, and so both of the terms had to be areas as well. She immediately spotted a problem with (A): the $b$ term under the radical was a length, not an area. On the other hand, the second term under the radical in answer (E) overshot the target, since $4bc$ was a multiple of a length times an area, not just a single area itself.

Three possibilities remained:

(B) $$x = \frac{-b \pm \sqrt{c - 4b^2}}{2a}$$

(F) $$x = \frac{-b \pm \sqrt{4ac - ab^2}}{2a}$$

(G) $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sadly, all of them made sense as expressions for lengths. Was there a way to separate them? Time on the clock was getting short.

In a flash of insight—possibly brought on by a hint of desperation—Anna noticed that there might be another way to look at the geometry. The first term in the quadratic equation involved multiplying together *three* quantities: $a$, $x$, and $x$ again. What if the quadratic equation were an equation about *volumes* instead of areas? Then $a$ would not be a number of squares, but instead it would be a length: the height of a box whose base was a square with sides of length $x$. That would make $b$ an area instead of a length, and $c$ a volume. Would that help?

The formula still needed to give a length for $x$, but now the $2a$ in the denominator was also a length. That meant that this time the numerator had to be an area: area divided by length gives length. In each of the remaining possibilities, the first term $-b$ was indeed an area, so that was encouraging—although unfortunately it didn't rule out any of them, either. But what about the square root? Since the value of the square root had to be an area, the quantity under the radical had to be the square of an area. Anna had no idea what the square of an area would represent, but it sounded four-dimensional. In that case each term under the square root should be the fourth power of a length, or an area times two lengths, or an area times an area, or a length times a volume. Did that eliminate any of the possibilities?

Good news! Option (B) had that lonely $c$ as the first term under the radical, which was just a volume, not a four-dimensional whatever, so it was out. (Anna wondered briefly what the square root of a volume might tell you, but there was no time to ponder that now.) And possibility (F) was even worse—the term $ab^2$ would be a *five*-dimensional quantity. She didn't want to think about what a five-dimensional world might look like. Fortunately, though, she did not need to, because there was only one possibility left. The last remaining formula had no problem with its dimensions, in terms of either areas or volumes.

With just a few seconds left on the clock, Anna leaned back in her chair. "The correct formula is (G)."

"You sound very confident," Nigel said. "Are you sure that's your final answer?"

*Always think about the units before you submit your final answer.* In one last moment of insight, Anna realized that she had been wrong about one thing: in fact there *were* units attached to this problem. Those units were *dollars*, and thanks to her dimensional analysis of the quadratic formula, she was about to win one hundred thousand of them.

"Yes, Nigel," she said with a smile. "That's my final answer."